Vector Continued Fractions*

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ABSTRACT

By exploiting an isomorphism between vectors and certain matrices, the theory of vector-valued continued fractions is developed as a special case of the theory of matrix-valued continued fractions. It is shown that if a given power series has coefficient vectors lying in Hilbert space, then the vector-valued Padé quotients derived from this series also lie in Hilbert space. Properties of the matrices occurring in the vector-matrix isomorphism are examined; in particular, it is shown that the numerical values of certain norms of a vector are equal to those of the corresponding norms of its companion matrix. The concept of a functional Padé table is introduced, and one of its properties is derived.

1. INTRODUCTION

In this paper we construct the formal theory underlying a numerical process for accelerating the convergence of vector sequences. Such sequences occur in the iterative solution of linear algebraic equations; they also arise quite naturally in numerical analysis in the following way: we are concerned with a function $S(x)$ which is defined on a certain interval of the x axis; $S(x)$ satisfies, for example, a differential or integral equation which we wish to solve; we discretize the problem and seek a vector $\mathbf{s}$ of solution values to the corresponding discrete problem which are defined at points of the x axis; we solve the discrete problem iteratively by selecting an initial approximation $\mathbf{s}_0$ to $\mathbf{s}$ and using a recursive scheme

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of the form

$$\mathbf{s}_{m+1} = \phi(\mathbf{s}_m) \qquad (m = 0, 1, \ldots),$$

to obtain a sequence of vectors $\{\mathbf{s}_m\}$ which, it is hoped, tends to $\mathbf{s}$.

The numerical convergence of a vector sequence $\{\mathbf{s}_m\}$ may be accelerated by means of the vector ε algorithm [1–3]. This algorithm entails the construction of a double sequence of vectors $\{\boldsymbol{\epsilon}_r^{(m)}\}$ which may be set in a two-dimensional array, the vector ε array, in which the suffix r denotes

$$\begin{array}{ccccc}
 & \boldsymbol{\epsilon}_0^{(0)} & & & \\
\boldsymbol{\epsilon}_{-1}^{(1)} & & \boldsymbol{\epsilon}_1^{(0)} & & \\
 & \boldsymbol{\epsilon}_0^{(1)} & & \boldsymbol{\epsilon}_2^{(0)} & \\
\boldsymbol{\epsilon}_{-1}^{(2)} & & \boldsymbol{\epsilon}_1^{(1)} & & \boldsymbol{\epsilon}_3^{(0)} \\
 & \boldsymbol{\epsilon}_0^{(2)} & & \boldsymbol{\epsilon}_2^{(1)} & \vdots \\
\boldsymbol{\epsilon}_{-1}^{(3)} & & \boldsymbol{\epsilon}_1^{(2)} & \vdots & \\
\vdots & \boldsymbol{\epsilon}_0^{(3)} & \vdots & & \\
 & \vdots & & &
\end{array}$$

FIG. 1.

a column and the superscript m a diagonal (Fig. 1). The initial conditions for the construction of this array are

$$\boldsymbol{\epsilon}_{-1}^{(m)} = \mathbf{0} \quad (m = 1, 2, \ldots), \qquad \boldsymbol{\epsilon}_0^{(m)} = \mathbf{s}_m \quad (m = 0, 1, \ldots); \tag{1}$$

the vectors of formulas (1) are to be found in the first two columns of the ε array. The remaining members of the ε array are constructed by a simple recursive process involving sums, differences, and inverses of vectors. The sum and difference of two vectors are defined by componentwise addition and subtraction, respectively. The inverse of the complex valued vector

$$\mathbf{z} = (z_1, z_2, \ldots, z_n) \tag{2}$$

is taken to be

$$\mathbf{z}^{-1} = \left(\sum_{t=1}^{n} z_t \bar{z}_t \right)^{-1} \bar{\mathbf{z}}, \tag{3}$$

where the bar denotes a complex conjugate. The relationship used in the construction of the vector ε array is

$$\boldsymbol{\epsilon}_{r+1}^{(m)} = \boldsymbol{\epsilon}_{r-1}^{(m+1)} + (\boldsymbol{\epsilon}_r^{(m+1)} - \boldsymbol{\epsilon}_r^{(m)})^{-1}. \tag{4}$$

Formula (4) concerns vectors lying at the four vertices of a lozenge in the ε array (Fig. 2). The extreme right-hand member of this group is

$$\boldsymbol{\epsilon}_r^{(m)}$$

$$\boldsymbol{\epsilon}_{r-1}^{(m+1)} \qquad\qquad \boldsymbol{\epsilon}_{r+1}^{(m)}$$

$$\boldsymbol{\epsilon}_r^{(m+1)}$$

FIG. 2.

expressed in terms of the remaining three. The vector ε array is built up, column by column from left to right, by systematic use of formula (4) with $r = 0, 1, \ldots$; $m = 0, 1, \ldots$. It is a fact of considerable numerical experience that a vector $\boldsymbol{\epsilon}_{2r}^{(m)}$, found in an even order column of the ε array, provides a considerably better approximation to the limit $\mathbf{s}$ of the sequence $\{\mathbf{s}_m\}$ than do any of the members of this sequence from which the vector $\boldsymbol{\epsilon}_{2r}^{(m)}$ in question is derived.

In order to give the reader some idea of the power of this convergence acceleration technique, we display some numerical results concerning the numerical solution of an integral equation [2, 3] by means of discretization and iteration as described above. The details concerning the particular integral equation, the discretization interval, and the numerical integration scheme used do not concern us here; it suffices to say that in this case the solution vector $\mathbf{s}$ to the discrete problem can be obtained with relative ease in another way; thus the various distances $||\boldsymbol{\epsilon}_{2r}^{(m)} - \mathbf{s}||$ (again the

TABLE I

	r			
m	0	2	4	6
0	5.41101			
1	3.85450	0.50711		
2	2.69179	0.04957	0.01263	
3	1.89641	0.01810	0.00203	0.00031
4	1.36144	0.00912	0.00042	
5	0.99621	0.00508		
6	0.73976			

definition of this distance does not concern us here) could be measured. In Table I we display a number of these distances. The successive distances given in the first column indicate that the original iterative scheme converges slowly; however, as the distance $\|\boldsymbol{\epsilon}_6^{(0)} - \mathbf{s}\|$ indicates, the transformed vector $\boldsymbol{\epsilon}_6^{(0)}$ offers a considerably better approximation to $\mathbf{s}$ than is provided in this case by any of the vectors $\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_6$; indeed it is possible to show that in the example under consideration the original iterative cycle must be repeated somewhat in excess of 8000 times before the resulting iterated vectors attain the same degree of approximation to $\mathbf{s}$ as is offered by $\boldsymbol{\epsilon}_6^{(0)}$.

It is clear from this example, and from many others for which the numerical results are similar, that the vector ε algorithm is a formidable resource of numerical analysis. However, the vector ε algorithm not only lacks a rigorous convergence theory (this is true of many methods of numerical analysis, and should not worry anyone) but also, at least until now, has lacked even the support of a formal theory. The vector ε algorithm is an analog of the scalar ε algorithm whose theory is based upon a concept, the continued fraction, for which no vector equivalent at present exists. In the following we begin, at least, to construct the formal apparatus of a theory of vector continued fractions.

2. CONTINUED FRACTIONS DERIVED FROM POWER SERIES

We begin with a brief review of some known facts concerning continued fractions, in particular continued fractions derived from power series.

The continued fraction

$$b_0 + \frac{a_1}{b_1 +} \cdots \frac{a_r}{b_r +} \cdots$$

has a meaning made clear by the notation employed: it is the limit, should this exist, of the convergents $\{C_r\}$, where

$$C_r = b_0 + \frac{a_1}{b_1 +} \cdots \frac{a_{r-1}}{b_{r-1} +} \frac{a_r}{b_r} \qquad (r = 1, 2, \ldots).$$

C_r may be computed by a process of successive division and addition starting with the partial quotient a_r/b_r, or more precisely

$$C_r = D_r \qquad (r = 1, 2, \ldots),$$

where

$$D_0 = b_r, \qquad D_{t+1} = b_{r-t-1} + \frac{a_{r-t}}{D_t} \qquad (t = 0, 1, \ldots, r-1). \tag{5}$$

Given a power series

$$F(\lambda) = \sum_{t=0}^{\infty} u_t \lambda^{-t-1},$$

it is formally possible to determine the coefficients $\{\alpha_t\}$, $\{\beta_t\}$ in the continued fraction

$$\frac{u_0}{\lambda - \alpha_0 -}\ \frac{\beta_0}{\lambda - \alpha_1 -} \cdots \frac{\beta_{r-2}}{\lambda - \alpha_{r-1} -} \cdots \tag{6}$$

by imposing the condition that the rth convergent

$$\frac{u_0}{\lambda - \alpha_0 -}\ \frac{\beta_0}{\lambda - \alpha_1 -} \cdots \frac{\beta_{r-2}}{\lambda - \alpha_{r-1}}$$

(which is, of course, a rational function of λ) of (6) has a series expansion in inverse powers of λ which agrees with $F(\lambda)$ as far as the first $2r$ terms; (6) is called the continued fraction associated with $F(\lambda)$ [4, Vol. II, Chapt. II]. It is easily shown that this convergent has the form $q_r(\lambda)/p_r(\lambda)$ where $p_r(\lambda)$ and $q_r(\lambda)$ are polynomials of degree r and $r-1$, respectively:

$$p_r(\lambda) = \sum_{t=0}^{r} k_{r,t} \lambda^t, \qquad q_r(\lambda) = \sum_{t=0}^{r-1} l_{r,t} \lambda^t \qquad (r = 0, 1, \ldots). \tag{7}$$

Thus from the definition of the associated expansion we have

$$\frac{q_r(\lambda)}{p_r(\lambda)} \sim \sum_{t=0}^{2r-1} u_t \lambda^{-t-1} + \sum_{t=2r}^{\infty} u_{r,t} \lambda^{-t-1}, \tag{8}$$

where in general $u_{r,t} \neq u_t$ $(t = 2r, 2r+1, \ldots)$. For example, we have

$$p_1(\lambda) = \lambda - \frac{u_1}{u_0}, \qquad q_1(\lambda) = u_0, \tag{9}$$

and

$$\frac{q_1(\lambda)}{p_1(\lambda)} \sim u_0 \lambda^{-1} + u_1 \lambda^{-2} + \text{terms in } \lambda^{-3}, \lambda^{-4}, \ldots. \tag{10}$$

The successive convergents $\{q_r(\lambda)/p_r(\lambda)\}$ provide a sequence of rational functions which approximate the sum or formal sum of $F(\lambda)$. It is a remarkable and at the same time extremely important fact that an associated continued fraction often converges far more rapidly than the series from which it is derived, and in certain cases converges even when the original series diverges.

Naturally an associated expansion may formally be derived from the delayed power series

$$F_m(\lambda) = \sum_{t=0}^{\infty} u_{m+t}\lambda^{-t-1};$$

the rth convergent of this expansion may be written as the quotient $q_r^{(m)}(\lambda)/p_r^{(m)}(\lambda)$ of two polynomials, again of degrees r and $r-1$, respectively. We have, as extensions to Eqs. (9) and (10),

$$p_1^{(m)}(\lambda) = \lambda - \frac{u_{m+1}}{u_m}, \qquad q_1^{(m)}(\lambda) = u_m$$

and

$$\frac{q_1^{(m)}(\lambda)}{p_1^{(m)}(\lambda)} \sim u_m\lambda^{-1} + u_{m+1}\lambda^{-2} + \text{terms in } \lambda^{-3}, \lambda^{-4}, \ldots .$$

The quotients $\{q_r^{(m)}(\lambda)/p_r^{(m)}(\lambda)\}$ provide a sequence of approximations to the sum or formal sum of $F_m(\lambda)$.

The relationship between the series $F(\lambda)$ and $F_m(\lambda)$ is clearly

$$F(\lambda) = \sum_{t=0}^{m-1} u_t\lambda^{-t-1} + \lambda^{-m}F_m(\lambda).$$

Thus the various expressions

$$\sum_{t=0}^{m-1} u_t\lambda^{-t-1} + \lambda^{-m}\frac{q_r^{(m)}(\lambda)}{p_r^{(m)}(\lambda)} \qquad (r = 0, 1, \ldots; \quad m = 0, 1, \ldots) \tag{11}$$

provide a sequence of approximations to the sum or formal sum of $F(\lambda)$.

Multiplying relationships (8) throughout by $p_r(\lambda)$, and replacing both $p_r(\lambda)$ and $q_r(\lambda)$ by their expressions (7), we have

$$\sum_{t=0}^{r-1} l_{r,t}\lambda^t \sim \left\{\sum_{t=0}^{r} k_{r,t}\lambda^t\right\}\left\{\sum_{t=0}^{2r-1} u_t\lambda^{-t-1} + \sum_{t=2r}^{\infty} u_{r,t}\lambda^{-t-1}\right\}. \tag{12}$$

If we examine the coefficients of $\lambda^{-1}, \lambda^{-2}, \ldots, \lambda^{-r-1}$ in relationship (12), we see that in general

$$\sum_{t=0}^{r} k_{r,t}u_{t+\tau}\begin{cases} = 0 & (\tau = 0, 1, \ldots, r-1), \\ \neq 0 & (\tau = r). \end{cases}$$

Thus the polynomials $\{p_r(\lambda)\}$ form an orthogonal system with respect to the system of moments $\{u_t\}$ (see, for example, [5, Vol. II, Chapt. X]), in the algebraic sense that the vector $(k_{r,0}, k_{r,1}, \ldots, k_{r,r})$ is orthogonal to the system of vectors $(u_\tau, u_{\tau+1}, \ldots, u_{\tau+r})$ $(\tau = 0, 1, \ldots, r-1)$; the polynomials $\{q_r(\lambda)\}$ are the associated orthogonal polynomials. $p_r^{(m)}(\lambda)$ is the orthogonal polynomial of order m and degree r; $q_r^{(m)}(\lambda)$ is its associated orthogonal polynomial.

3. THE PADÉ TABLE [6]

Given a power series

$$f(\mu) = \sum_{t=0}^{\infty} u_t \mu^t$$

it is formally possible [4, Vol. II, Chapt. VI] to construct a certain double sequence $\rho_{i,j}$ $(i = 0, 1, \ldots; j = 0, 1, \ldots)$ of rational functions of μ. The numerator polynomial of $\rho_{i,j}$ is of the jth degree, the denominator polynomial of degree i; this quotient is characterized by the property that its series expansion in ascending powers of μ agrees with the series $f(\mu)$ as far as the term in μ^{i+j}. Specifically

$$\rho_{i,j} = \frac{\sum_{t=0}^{j} \pi_{i,j,t}\mu^t}{\sum_{t=0}^{i} \pi'_{i,j,t}\mu^t} \sim \sum_{t=0}^{i+j} u_t\mu^t + \sum_{t=i+j+1}^{\infty} u_{i,j,t}\mu^t \quad (i = 0, 1, \ldots; j = 0, 1, \ldots), \tag{13}$$

where in general $u_{i,j,t} \neq u_t$ $(t = i + j + 1, i + j + 2, \ldots)$.

The functions $\{\rho_{i,j}\}$ may be set in a two-dimensional array in which the suffix i indicates a row number and the second suffix j a column number; this is the Padé table.

It is clear that Eqs. (8) and (13) are effectively formulations of the same property; the quotients lying on and above the principal diagonal of the Padé table are simply related to the rational expressions of the form (11); indeed we have

$$\sum_{t=0}^{m-1} u_t\lambda^{-t-1} + \lambda^{-m}\frac{q_r^{(m)}(\lambda)}{p_r^{(m)}(\lambda)} = \mu\rho_{r,m+r}, \qquad \mu\lambda = 1$$

$$(r = 0, 1, \ldots; m = 0, 1, \ldots).$$

For the sake of completeness we remark that one can derive a relationship concerning associated continued fractions and the quotients of the lower half of the Padé table. The reciprocal series

$$F^*(\lambda) = \sum_{t=0}^{\infty} u_t^* \lambda^{-t-1}$$

is derived from the series $F(\lambda)$ by means of the formal relationship

$$F(\lambda) = \frac{u_0}{\lambda\{1 + F^*(\lambda)\}}.$$

The delayed series

$$F_m^*(\lambda) = \sum_{t=0}^{\infty} u_{m+t}^* \lambda^{-t-1}$$

obtained from $F^*(\lambda)$ has an associated continued fraction which we denote by

$$\frac{u_m^*}{\lambda - \alpha_0^{(m)*} -} \frac{\beta_0^{(m)*}}{\lambda - \alpha_1^{(m)*} -} \cdots \frac{\beta_{r-2}^{(m)*}}{\lambda - \alpha_{r-1}^{(m)*} -} \cdots.$$

The successive convergents of this expansion are denoted by $q_r^{(m)*}(\lambda)/p_r^{(m)*}(\lambda)$ $(r = 0, 1, \ldots)$. We then have

$$\frac{u_0}{1 + \sum_{t=0}^{m-1} u_t^* \lambda^{-t-1} + \lambda^{-m}(q_r^{(m)*}(\lambda)/p_r^{(m)*}(\lambda))} = \rho_{r+m,r}, \qquad \mu\lambda = 1$$

$$(r = 0, 1, \ldots; m = 0, 1, \ldots).$$

Use of the series $f(\mu)$ leads most directly to the concept of the Padé quotient, and facilitates the most unified presentation of the formal theory of the Padé table. It must be remarked, however, without quoting any special results in detail, that most of the results in the convergence theory of the Padé table relate to the diagonal sequences. They are in essence convergence results in the theory of associated continued fractions and depend to a large extent upon classical properties of orthogonal polynomials. Thus it transpires that use of series $F(\lambda)$ and $F_m(\lambda)$ and the consequent formalism is the most suitable for the theoretical investigation of the Padé table.

4. THE EPSILON ALGORITHM [7]

The rational functions (11) and indeed the whole of the Padé table may be constructed with the help of a very simple recursive process. In the first case, we have [8]

LEMMA 1. *If functions* $\varepsilon_r^{(m)}$ $(r = 0, 1, \ldots; m = 0, 1, \ldots)$ *are constructed by application of the relationships*

$$\varepsilon_{r+1}^{(m)} = \varepsilon_{r-1}^{(m+1)} + (\varepsilon_r^{(m+1)} - \varepsilon_r^{(m)})^{-1} \tag{14}$$

with $r = 0, 1, \ldots; m = 0, 1, \ldots$ *to the initial values*

$$\varepsilon_{-1}^{(m)} = 0 \quad (m = 1, 2, \ldots), \qquad \varepsilon_0^{(m)} = \sum_{t=0}^{m-1} u_t \lambda^{-t-1} \quad (m = 0, 1, \ldots),$$

then

$$\varepsilon_{2r}^{(m)} = \sum_{t=0}^{m-1} u_t \lambda^{-t-1} + \lambda^{-m} \frac{q_r^{(m)}(\lambda)}{p_r^{(m)}(\lambda)} \qquad (r = 0, 1, \ldots; m = 0, 1, \ldots).$$

In the second case we have [9]

LEMMA 2. *If functions* $\varepsilon_r^{(m)}$ $(r = 0, 1, \ldots;\ m = -(r \div 2) - 1, -(r \div 2), \ldots)$ *are constructed by application of the relationships* (14) *with* $r = 0, 1, \ldots;$ $m = -(r \div 2) - 1, -(r \div 2), \ldots$ *to the initial values*

$$\varepsilon_{-1}^{(m)} = 0 \quad (m = 0, 1, \ldots), \qquad \varepsilon_{2m}^{(-m-1)} = 0 \quad (m = 0, 1, \ldots),$$

$$\varepsilon_0^{(m)} = \sum_{t=0}^{m} u_t \mu^t \qquad (m = 0, 1, \ldots),$$

then

$$\varepsilon_{2r}^{(m)} = \rho_{r,m+r} \qquad (r = 0, 1, \ldots; m = -r, -r+1, \ldots).$$

For the sake of clarity, we point out that for fixed r the functions $\varepsilon_{2r}^{(m)}$ $(m = -r, -r+1, \ldots)$ lie in a column of the ε array, and the functions $\rho_{r,m+r}$ lie in a row of the Padé table.

The equations of Lemmas 1 and 2 may be interpreted in two ways. First, they may be considered to represent relationships between rational functions: Eq. (14) is in this case a prescription for constructing the coefficients in the numerator and denominator of such a function. Second, the value of the variable λ or μ may be considered to be fixed; Eq. (14) is then an arithmetic relationship and may be used recursively to compute the values of the above-mentioned rational functions.

If two neighboring members, $\varepsilon_r^{(m+1)}$ and $\varepsilon_r^{(m)}$, of the ε array are equal, then the ε algorithm breaks down in the sense that a sector of the ε array cannot be constructed (if $\varepsilon_r^{(m+1)}$ and $\varepsilon_r^{(m)}$ are both infinite this sector has its vertex at $\varepsilon_{r+1}^{(m)}$; otherwise at $\varepsilon_{r+3}^{(m-1)}$). Certain singular rules enabling this breakdown to be repaired have been devised [10]. In the first of the implementations described in the preceding paragraph, breakdown occurs when $\varepsilon_r^{(m+1)}$ and $\varepsilon_r^{(m)}$ represent identically equal rational functions; in the second implementation, breakdown occurs when numerical values of $\varepsilon_r^{(m+1)}$ and $\varepsilon_r^{(m)}$ are fortuitously equal. For the sake of completeness we mention that if certain assumptions concerning the $\{u_t\}$ are made, then breakdown in the first implementation cannot occur, and that if additional restrictions are imposed upon λ or μ, breakdown in the second implementation is also impossible.

In practice, the ε algorithm provides a very effective convergence acceleration technique. Given a slowly convergent or divergent sequence $\{S_m\}$, relationships (14) are applied to the initial values

$$\varepsilon_{-1}^{(m)} = 0 \quad (m = 1, 2, \ldots), \qquad \varepsilon_0^{(m)} = S_m \quad (m = 0, 1, \ldots);$$

it is a fact of computational experience that a derived quantity $\varepsilon_{2r}^{(m)}$ often approximates the limit or formal limit of the original sequence far more accurately than do the members of the sequence from which $\varepsilon_{2r}^{(m)}$ is derived. The use of this acceleration technique may be given a theoretical justification by selecting a variable λ and thus determining a set of coefficients $\{u_t\}$ such that

$$\sum_{t=0}^{m-1} u_t \lambda^{-t-1} = S_m \qquad (m = 0, 1, \ldots);$$

the convergence acceleration properties of the ε algorithm may then, in any given case, be based on the theory of continued fractions.

The results of Lemmas 1 and 2 were first proved by a direct appeal to determinantal expressions for the functions involved; however, a purely constructive proof has recently been given. It belongs to the domain of classical theory that the coefficients $\{\alpha_t^{(m)}\}$, $\{\beta_t^{(m)}\}$ of the various associated expansions, and also the polynomials $\{p_r^{(m)}(\lambda)\}$, $\{q_r^{(m)}(\lambda)\}$, can be constructed by means of simple recursive algorithms [11, 12]. It was the service of F. L. Bauer [13] to reveal the existence of a bridge of algorithms stretching from these classical recursions to the ε algorithm; each algorithm is constructed from its predecessor by direct substitution and simple manipulations; the theory of determinants can wholly be dispensed with. The existence of this constructive proof was a crucial factor in later developments which we shall shortly describe.

5. NONCOMMUTATIVE CONTINUED FRACTIONS

Recently [14] an extensive theory of noncommutative continued fractions has been developed: this has led to the establishment of a noncommutative version of the quotient-difference algorithm, to a theory of noncommutative orthogonal polynomials, to an extension of the theory of the ε algorithm which concerns quantities satisfying a noncommutative law of multiplication, and finally to the concept of what may with complete propriety be called an operator-valued Padé table.

It is assumed that the coefficients of the continued fractions, the convergents, and indeed all the auxiliary objects which occur in the theory are elements of a division ring $\mathfrak{N}$: the elements $\mathbf{N}_0, \mathbf{N}_1, \mathbf{N}_2, \ldots$ of $\mathfrak{N}$ satisfy the following assumptions:

(1) Addition is commutative and associative: for every pair $\mathbf{N}_i, \mathbf{N}_j \in \mathfrak{N}$ there exists an element $\mathbf{N}_k$ such that

$$\mathbf{N}_i + \mathbf{N}_j = \mathbf{N}_j + \mathbf{N}_i = \mathbf{N}_k;$$

furthermore for every trio $\mathbf{N}_r, \mathbf{N}_s, \mathbf{N}_t \in \mathfrak{N}$

$$(\mathbf{N}_r + \mathbf{N}_s) + \mathbf{N}_t = \mathbf{N}_r + (\mathbf{N}_s + \mathbf{N}_t).$$

(2) Multiplication is associative, distributive, and in general noncommutative: for every pair $\mathbf{N}_i, \mathbf{N}_j \in \mathfrak{N}$ there exists an element $\mathbf{N}_k \in \mathfrak{N}$ such that

$$\mathbf{N}_i\mathbf{N}_j = \mathbf{N}_k;$$

furthermore for every trio $\mathbf{N}_r, \mathbf{N}_s, \mathbf{N}_t \in \mathfrak{N}$

$$(\mathbf{N}_r\mathbf{N}_s)\mathbf{N}_t = \mathbf{N}_r(\mathbf{N}_s\mathbf{N}_t);$$

also

$$\mathbf{N}_r(\mathbf{N}_s + \mathbf{N}_t) = \mathbf{N}_r\mathbf{N}_s + \mathbf{N}_r\mathbf{N}_t;$$

in general

$$\mathbf{N}_i\mathbf{N}_j \neq \mathbf{N}_j\mathbf{N}_i.$$

(3) There exists a subset of scalars $\mathfrak{S} \in \mathfrak{N}$ which in themselves constitute a field; for every $\mathbf{N}_i \in \mathfrak{S}$ and $\mathbf{N}_j \in \mathfrak{N}$

$$\mathbf{N}_i\mathbf{N}_j = \mathbf{N}_j\mathbf{N}_i;$$

in particular **I** and **0** are the unit and zero elements respectively of $\mathfrak{S}$.

(4) There exists a subset $\mathfrak{P}$ of $\mathfrak{N}$ of invertible elements; to every element $\mathbf{N} \in \mathfrak{P}$ there corresponds an element $\mathbf{N}^{-1} \in \mathfrak{P}$ for which

$$\mathbf{N}\mathbf{N}^{-1} = \mathbf{N}^{-1}\mathbf{N} = \mathbf{I}.$$

If the coefficients of a continued fraction satisfy a noncommutative law of multiplication, then it is essential to specify the order in which multiplication by the inverse takes place in relationships of the form (5). Two systems of continued fractions have been studied: The *pre*-continued fraction

$$pre\left[\mathbf{B}_0 + \frac{\mathbf{A}_1}{\mathbf{B}_1 +}\ \frac{\mathbf{A}_2}{\mathbf{B}_2 +} \cdots \frac{\mathbf{A}_r}{\mathbf{B}_r +} \cdots\right]$$

has successive convergents $\{_{pre}\mathbf{C}_r\}$ given by

$$_{pre}\mathbf{C}_r = \mathbf{D}_r' \qquad (r = 1, 2, \ldots),$$

where

$$\mathbf{D}_0' = \mathbf{B}_r, \qquad \mathbf{D}_{t+1}' = \mathbf{B}_{r-t-1} + \mathbf{D}_t'^{-1}\mathbf{A}_{r-t} \qquad (t = 0, 1, \ldots, r-1);$$

the *post*-continued fraction has successive convergents $\{_{post}\mathbf{C}_r\}$ given by

$$_{post}\mathbf{C}_r = \mathbf{D}_r'',$$

where

$$\mathbf{D}_0'' = \mathbf{B}_r, \qquad \mathbf{D}_{t+1}'' = \mathbf{B}_{r-t-1} + \mathbf{A}_{r-t}\mathbf{D}_t''^{-1} \qquad (t = 0, 1, \ldots, r-1).$$

In general both the value and the successive convergents of the *pre*-continued fraction with coefficients $\{\mathbf{A}_t\}$, $\{\mathbf{B}_t\}$ differ from those of the *post*-continued fraction with the same coefficients. For example,

$$pre\left[\mathbf{B}_0 + \frac{\mathbf{A}_1}{\mathbf{B}_1}\right] = \mathbf{B}_0 + \mathbf{B}_1^{-1}\mathbf{A}_1, \qquad post\left[\mathbf{B}_0 + \frac{\mathbf{A}_1}{\mathbf{B}_1}\right] = \mathbf{B}_0 + \mathbf{A}_1\mathbf{B}_1^{-1}.$$

It has been possible to develop the theory of continued fractions of both systems associated with a power series $\sum_{t=0}^{\infty} \mathbf{U}_t\lambda^{-t-1}$ whose argument λ is scalar and whose coefficients $\{\mathbf{U}_t\}$ obey a noncommutative law of multiplication; the coefficients of the continued fractions of the two systems, derived from the same power series, in general differ.

It has also been possible to develop the theory of two systems of orthogonal polynomials $\{{}_{pre}\mathbf{P}_r^{(m)}(\lambda)\}$, $\{{}_{post}\mathbf{P}_r^{(m)}(\lambda)\}$ and their associated orthogonal polynomials $\{{}_{pre}\mathbf{Q}_r^{(m)}(\lambda)\}$, $\{{}_{post}\mathbf{Q}_r^{(m)}(\lambda)\}$ derived from a set of moments $\{\mathbf{U}_t\}$ which obey a noncommutative law of multiplication. The polynomials of the two systems are characterized by the property that the power series expansions of the quotients ${}_{pre}\mathbf{P}_r^{(m)}(\lambda)^{-1}\,{}_{pre}\mathbf{Q}_r^{(m)}(\lambda)$ and ${}_{post}\mathbf{Q}_r^{(m)}(\lambda)\,{}_{post}\mathbf{P}_r^{(m)}(\lambda)^{-1}$ agree with the series $\sum_{t=0}^{\infty}\mathbf{U}_t\lambda^{-t-1}$ as far as the first $2r$ terms. The coefficients in the polynomials of the two systems differ: for example,

$$
\begin{aligned}
{}_{pre}\mathbf{P}_1^{(m)}(\lambda) &= \lambda - \mathbf{U}_{m+1}\mathbf{U}_m^{-1}, & {}_{pre}\mathbf{Q}_1^{(m)}(\lambda) &= \mathbf{U}_m,\\
{}_{post}\mathbf{P}_1^{(m)}(\lambda) &= \lambda - \mathbf{U}_m^{-1}\mathbf{U}_{m+1}, & {}_{post}\mathbf{Q}_1^{(m)}(\lambda) &= \mathbf{U}_m;
\end{aligned}
$$

by chance, due to their simple construction ${}_{pre}\mathbf{Q}_1^{(m)}(\lambda)$ and ${}_{post}\mathbf{Q}_1^{(m)}(\lambda)$ are equal; but such an equivalence ceases to hold for polynomials of higher order. The reader will easily verify that

$$
\begin{aligned}
&{}_{pre}\mathbf{P}_1^{(m)}(\lambda)^{-1}\,{}_{pre}\mathbf{Q}_1^{(m)}(\lambda) \sim \mathbf{U}_m\lambda^{-1} + \mathbf{U}_{m+1}\lambda^{-2} + \text{terms in } \lambda^{-3}, \lambda^{-4}, \ldots\\
&{}_{post}\mathbf{Q}_1^{(m)}(\lambda)\,{}_{post}\mathbf{P}_1^{(m)}(\lambda)^{-1} \sim \mathbf{U}_m\lambda^{-1} + \mathbf{U}_{m+1}\lambda^{-2} + \text{terms in } \lambda^{-3}, \lambda^{-4}, \ldots .
\end{aligned}
\tag{15}
$$

It is possible to construct noncommutative versions both of the classical systems of recursions occurring in the theory of scalar associated continued fractions, and of Bauer's bridge leading from these recursions to the ε algorithm.

The relationships of the ε algorithm do not involve multiplication, of either the pre or post kind; thus, starting with differing initial conditions, and proceeding through differing systems of recursions, the theories of the associated continued fractions of the two systems culminate in an algorithm, the ε algorithm, which is common to both.

The end product of this formal theory is the following [14]:

LEMMA 3. *If the relationships*

$$\mathbf{E}_{r+1}^{(m)} = \mathbf{E}_{r-1}^{(m+1)} + (\mathbf{E}_r^{(m+1)} - \mathbf{E}_r^{(m)})^{-1} \tag{16}$$

with $r = 0, 1, \ldots; m = 0, 1, \ldots$ *can be applied to the initial conditions*

$$\mathbf{E}_{-1}^{(m)} = 0 \quad (m = 1, 2, \ldots), \qquad \mathbf{E}_0^{(m)} = \sum_{t=0}^{m-1} \mathbf{U}_t\lambda^{-t-1} \quad (m = 0, 1, \ldots), \tag{17}$$

where $\mathbf{U}_t \in \mathfrak{R}$ $(t = 0, 1, \ldots)$ *and* λ *is scalar, to produce operator-valued rational functions* $\{\mathbf{E}_r^{(m)}\}$, *then*

$$\mathbf{E}_{2r}^{(m)} = \sum_{t=0}^{m-1} \mathbf{U}_t \lambda^{-t-1} + \lambda^{-m} {}_{pre}\mathbf{P}_r^{(m)}(\lambda)^{-1} {}_{pre}\mathbf{Q}_r^{(m)}(\lambda) \tag{18}$$

$$(r = 0, 1, \ldots; m = 0, 1, \ldots);$$

if relationships (16) *can be applied to the same initial conditions* (17), *then*

$$\mathbf{E}_{2r}^{(m)} = \sum_{t=0}^{m-1} \mathbf{U}_t \lambda^{-t-1} + \lambda^{-m} {}_{post}\mathbf{Q}_r^{(m)}(\lambda) {}_{post}\mathbf{P}_r^{(m)}(\lambda)^{-1} \tag{19}$$

$$(r = 0, 1, \ldots; m = 0, 1, \ldots).$$

Since both the initial conditions (17) and the form of the ε algorithm relationships (16) are the same for the two systems of continued fractions, we have by equating expressions (18) and (19)

$${}_{pre}\mathbf{P}_r^{(m)}(\lambda)^{-1} {}_{pre}\mathbf{Q}_r^{(m)}(\lambda) = {}_{post}\mathbf{Q}_r^{(m)}(\lambda) {}_{post}\mathbf{P}_r^{(m)}(\lambda)^{-1} \tag{20}$$

$$(r = 0, 1, \ldots; m = 0, 1, \ldots).$$

If the reader cares to examine expressions (15) in detail, he will find that they are both equal to $(\lambda \mathbf{U}_m^{-1} - \mathbf{U}_m^{-1}\mathbf{U}_{m+1}\mathbf{U}_m^{-1})^{-1}$.

Although the orthogonal and associated orthogonal polynomials found by considering conditions of *pre*-orthogonality and *post*-orthogonality form quite distinct systems, their quotients, as indicated by Eq. (20), are identical.

In the context of ascending power series, we have [14]

LEMMA 4. *If the relationships* (16), *with* $r = 0, 1, \ldots; m = -(r \div 2) - 1, -(r \div 2), \ldots,$ *can be applied to the initial values*

$$\mathbf{E}_{-1}^{(m)} = 0 \quad (m = 0, 1, \ldots), \qquad \mathbf{E}_{2m}^{(-m-1)} \quad (m = 0, 1, \ldots),$$

$$\mathbf{E}_0^{(m)} = \sum_{t=0}^{m} \mathbf{U}_t \mu^t \qquad (m = 0, 1, \ldots), \tag{21}$$

where $\mathbf{U}_t \in \mathfrak{R}$ $(t = 0, 1, \ldots)$ *and* μ *is scalar, to produce operator valued rational functions* $\{\mathbf{E}_r^{(m)}\}$, *then*

$$\mathbf{E}_{2r}^{(m)} \sim \sum_{t=0}^{m+2r} \mathbf{U}_t \mu^t + \sum_{t=m+2r+1}^{\infty} \mathbf{U}_{m,r,t} \mu^t,$$

where in general $\mathbf{U}_{m,r,t} \neq \mathbf{U}_t$ $(t = m + 2r + 1, m + 2r + 2, \ldots)$.

There is no question of having a *pre-* or *post*-Padé table: the operator valued Padé table is unique.

It should be emphasized that all of the formal theory of operator valued continued fractions has been established by the consistent use of nonlinear recursions, and simple manipulations involving the operators related by these recursions. Although the concept of a determinant with noncommutatively multiplying elements has been introduced (for example, see [16–22]; for related matters, see [23–29]), the higher flights of the theory of such determinants have not been ventured upon; it was thus impossible, when constructing the theory of the operator valued Padé table, to adapt the many determinantal results in the theory of scalar continued fractions; whether analogs of these determinantal results can in fact be constructed is a matter awaiting investigation.

Again, just as in the scalar case, the operator ε algorithm provides a powerful acceleration technique for, for example, a square matrix sequence $\{\mathbf{S}_m\}$ (such as occurs in the iterative treatment of a partial differential equation whose solution is required at points lying on a square grid). Relationships (16) with $r = 0, 1, \ldots; m = 0, 1, \ldots,$ are applied to the initial values

$$\mathbf{E}_{-1}^{(m)} = \mathbf{0} \quad (m = 1, 2, \ldots), \qquad \mathbf{E}_0^{(m)} = \mathbf{S}_m \quad (m = 0, 1, \ldots); \tag{22}$$

a transformed matrix is often a substantially better approximation to the limit or formal limit of $\{\mathbf{S}_m\}$ than any of the original matrices from which it is derived.

6. THE VECTOR EPSILON ALGORITHM

Although the convergence acceleration properties of the operator ε algorithm are indeed remarkable, it must be stated that its scope of application is severely limited (the algorithm in this form deals only with square matrices and other arrays of a similarly restricted form); furthermore this acceleration procedure is computationally very expensive, in view of the matrix inversion involved at each stage of the calculations. A substantial increase in the effectiveness of the ε algorithm resulted from a suggestion of Prof. K. Samelson that use should be made of the vector inverse described in the introduction: the inverse $\mathbf{z}^{-1}$ of the vector $\mathbf{z}$ of Eq. (2) is given by Eq. (3).

We remark parenthetically concerning this definition that it is an eminently reasonable interpretation of the inverse of a vector. If the vector has but one component, then $\mathbf{z}^{-1}$ reduces to the reciprocal of the complex number $\mathbf{z}$. If $\mathbf{z}$ has three real components x, y, z then Eq. (3) gives

$$(x, y, z)^{-1} = \left(\frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right);$$

this is the birational transformation of Cremona [30]. More generally $\mathbf{z}^{-1}$ is the inverse point of $\mathbf{z}$ with respect to the unit sphere in n dimensions. Finally it may be shown that $\mathbf{z}^{-1}$ is the transpose of the generalized inverse of a rectangular matrix of Moore [31] and Penrose [32].

From the viewpoint of application, we remark first that any array of numbers occurring in practical numerical analysis can be mapped onto a vector, and second that the computational cost of using vector inversion is relatively low.

This vector inverse was incorporated into the ε algorithm relationships in a manner described and illustrated in the introduction; after a number of successful numerical experiments, the vector ε algorithm became established as an acceleration technique of general application and great power.

It proved to be extremely difficult to establish a mathematical theory of the vector ε algorithm; the root of the difficulties appears to lie in the simple fact that vectors are not closed with respect to multiplication. Unsuccessful attempts were made both to derive a direct representation of the vectors $\{\boldsymbol{\epsilon}_r^{(m)}\}$ in terms of the vectors $\{\mathbf{s}_m\}$ from which they are derived, and to relate the vectors $\{\boldsymbol{\epsilon}_r^{(m)}\}$ to vectors produced by other recursive schemes. Clearly an oblique approach to this theory was called for.

As a result of a series of numerical experiments, and of theoretical investigations concerning vector sequences of certain special types, the author was led to propose the following conjecture [33]: *if n-dimensional complex-valued vectors $\{\boldsymbol{\epsilon}_r^{(m)}\}$ can be constructed by means of the relationships* (3) *from the initial values* (1) *and an irreducible linear recursion*

$$\sum_{t=0}^{h} c_t \mathbf{s}_{m+t} = \left(\sum_{t=0}^{h} c_t \right) \mathbf{s} \qquad (m = 0, 1, \ldots)$$

(*where the $\{c_t\}$ are complex numbers independent of m, and the $\{\mathbf{s}_t\}$ are n-dimensional complex-valued vectors*) *prevails among the vectors $\{\mathbf{s}_m\}$, then identically*

$$\boldsymbol{\epsilon}_{2h}^{(m)} = \mathbf{s} \qquad (m = 0, 1, \ldots).$$

It may appear to the reader that this conjecture is the sort of result that can be settled one way or the other without much trouble, but although it has been generally known in mathematical circles for some time it still remains to be proven in its full generality. As is the case with most mathematical conjectures, the end result is not especially significant; the importance of the conjecture lies in the mathematical theory which must be constructed for its proof.

7. McLEOD'S ISOMORPHISM

A fundamental breakthrough to a proof of the above conjecture in the case which the $\{c_t\}$ are real numbers was made by J. B. McLeod [**34**]. It can be shown that a version of the above result is true if the $\{\mathbf{s}_m\}$ and $\mathbf{s}$ are replaced by square matrices; if therefore it is possible to induce an isomorphism between real n-dimensional vectors $\mathbf{z}$ and certain matrices $\mathbf{Z}$ (we write $\mathbf{z} \leftrightarrow \mathbf{Z}$) such that sums, differences, and inverses of vectors correspond to sums, differences, and inverses of associated matrices, then a complete isomorphism is thereby induced between vector and matrix ε arrays: if the vector ε array is constructed from the vector sequence $\{\mathbf{s}_m\}$, and a matrix ε array from a matrix sequence $\{\mathbf{S}_m\}$, and furthermore $\mathbf{s}_m \leftrightarrow \mathbf{S}_m$ $(m = 0, 1, \ldots)$, then $\boldsymbol{\epsilon}_r^{(m)} \leftrightarrow \mathbf{E}_r^{(m)}$ $(r = 0, 1, \ldots; m = 0, 1, \ldots)$; the special case of the conjecture is proved.

Although the vector-matrix isomorphism of the preceding paragraph was introduced to prove a special conjecture, we see immediately that in a wider context it enables the general theory of the matrix ε algorithm to be mapped onto the theory of the vector ε algorithm.

A vector-matrix isomorphism of the required sort was determined by McLeod: he constructed a set of $2^n \times 2^n$ matrices $\{\boldsymbol{\Gamma}_t^{(n)}\}$ which satisfy the relationships

$$\begin{aligned} \boldsymbol{\Gamma}_t^{(n)2} &= \mathbf{I} \qquad (t = 1, 2, \ldots, n), \\ \boldsymbol{\Gamma}_t^{(n)}\boldsymbol{\Gamma}_{t'}^{(n)} + \boldsymbol{\Gamma}_{t'}^{(n)}\boldsymbol{\Gamma}_t^{(n)} &= \mathbf{0} \qquad (t = 1, 2, \ldots, n-1; t' = t+1, t+2, \ldots, n); \end{aligned} \tag{23}$$

the isomorphism between a real vector $\mathbf{z}$ of Eq. (2) and its associated matrix $\mathbf{Z}$ is given by

$$\mathbf{Z} = \sum_{t=1}^{n} z_t \boldsymbol{\Gamma}_t^{(n)}. \tag{24}$$

Clearly if $\mathbf{z}'$ and $\mathbf{z}''$ are two real n-dimensional vectors of the form (2), and their companion matrices in the sense of Eq. (24) are $\mathbf{Z}'$ and $\mathbf{Z}''$, respectively, then

$$\mathbf{z}' \pm \mathbf{z}'' \leftrightarrow \mathbf{Z}' \pm \mathbf{Z}''.$$

Furthermore, we obtain from Eq. (24)

$$\mathbf{Z}^2 = \sum_{t=1}^{n} z_t^2 \mathbf{\Gamma}_t^{(n)^2} + \sum_{t=1}^{n-1} \sum_{t'=t+1}^{n} z_t z_{t'} (\mathbf{\Gamma}_t^{(n)} \mathbf{\Gamma}_{t'}^{(n)} + \mathbf{\Gamma}_{t'}^{(n)} \mathbf{\Gamma}_t^{(n)})$$

$$= \left(\sum_{t=1}^{n} z_t^2 \right) \mathbf{I}$$

in consequence of Eqs. (23). Thus we have

$$\mathbf{Z}^{-1} = \left(\sum_{t=1}^{n} z_t^2 \right)^{-1} \mathbf{Z}$$

or, using the definition of Eq. (3) with real values of the components,

$$\mathbf{z}^{-1} \leftrightarrow \mathbf{Z}^{-1};$$

thus the isomorphism has the properties ascribed to it.

McLeod actually constructed a set of matrices which satisfy the required relationships. They are real and of dimension $2^n \times 2^n$; we denote the $2^t \times 2^t$ matrix of zeros by $\mathbf{0}^{(t)}$ and the $2^t \times 2^t$ unit matrix by $\mathbf{I}^{(t)}$; McLeod's representation is then as follows: If we write

$$\mathbf{M}_1^{(n)} = \begin{pmatrix} \mathbf{I}^{(n-1)} & \mathbf{0}^{(n-1)} \\ \mathbf{0}^{(n-1)} & -\mathbf{I}^{(n-1)} \end{pmatrix},$$

$$\mathbf{M}_2^{(n)} = \begin{pmatrix} \mathbf{0}^{(n-1)} & \begin{matrix} \mathbf{I}^{(n-2)} & \mathbf{0}^{(n-2)} \\ \mathbf{0}^{(n-2)} & -\mathbf{I}^{(n-2)} \end{matrix} \\ \begin{matrix} \mathbf{I}^{(n-2)} & \mathbf{0}^{(n-2)} \\ \mathbf{0}^{(n-2)} & -\mathbf{I}^{(n-2)} \end{matrix} & \mathbf{0}^{(n-1)} \end{pmatrix},$$

then each matrix $\mathbf{M}_t^{(n)}$ is obtained from its predecessor $\mathbf{M}_{t-1}^{(n)}$ by replacing the matrix $\mathbf{0}^{(n-t+1)}$ by the matrix

$$\begin{pmatrix} \mathbf{I}^{(n-t)} & \mathbf{0}^{(n-t)} \\ \mathbf{0}^{(n-t)} & -\mathbf{I}^{(n-t)} \end{pmatrix}$$

and annihilating the remaining elements. It is easily verified that these matrices satisfy Eqs. (23).

8. THE CLIFFORD ALGEBRA

For the sake of completeness we remark that numbers given by a relationship of the form (24) with scalar coefficients, in conjunction with Eqs. (23), are known as Clifford numbers [35] (see, for example, [36, 37]). They occur in quantum theory [38, 39] and in particular in Dirac's theory of electron spin [40, 41]; they are also extensively employed in the algebraic theory of spinors [42].

A number of representations of the $\{\mathbf{\Gamma}_t^{(n)}\}$ have been given; we mention the Cartan matrices [43] for the case $n = 3$, and the Dirac matrices for the case $n = 4$. Further work on the subject has been carried out by Eddington [44] and Newman [45]. Cartan [46] obtained representations for general positive integer values of n in terms of tensor products of Cartan matrices.

Hurwitz [47] gave a general theory of real matrices which satisfy relationships (23) for arbitrary positive integer values of n; McLeod's matrices are subsumed within this scheme.

9. A FURTHER SYSTEM OF ANTICOMMUTING MATRICES

Although the introduction of the Hurwitz-McLeod matrices was a crucial step in establishing the theory of the vector ε algorithm, with regard to further developments they suffer from two disadvantages. First, the isomorphism (24), as described, relates only to real-valued vectors. Second, the law of formation of any one of the matrices changes as n is replaced by $n + 1$; for example, the position of the nonzero element in the first row of $\mathbf{M}_1^{(n)}$ moves as n is increased; this is particularly unfortunate when we come to consider vectors with an infinite number of components.

As we shall see later, the first difficulty is easily disposed of. With regard to the second, the suggestion naturally prompts itself that if we put the zeros and units in other places, the individual matrix corresponding to a fixed component might, in a way which can be exploited, retain its structure. This suggestion proves to be a fruitful one, and as a preliminary we introduce

THEOREM 1. *Denote the $2^r \times 2^r$ matrix with units along the principal backward diagonal and zeros elsewhere by $\tilde{\mathbf{I}}^{(r)}$. The set of r matrices $\mathbf{K}_t^{(r)}$ $(t = 1, 2, \ldots, r)$, each of dimension $2^r \times 2^r$, is defined recursively as follows:*

$$\mathbf{K}_1^{(1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\mathbf{K}_t^{(r)} = \begin{pmatrix} \mathbf{K}_t^{(r-1)} & \mathbf{O}^{(r-1)} \\ \mathbf{O}^{(r-1)} & \mathbf{K}_t^{(r-1)} \end{pmatrix} \qquad (r = 2, 3, \ldots; t = 1, 2, \ldots, r-1),$$

$$\mathbf{K}_r^{(r)} = \begin{pmatrix} \mathbf{O}^{(r-1)} & \tilde{\mathbf{I}}^{(r-1)} \\ -\tilde{\mathbf{I}}^{(r-1)} & \mathbf{O}^{(r-1)} \end{pmatrix} \qquad (r = 2, 3, \ldots);$$

they satisfy the relationships

$$\mathbf{K}_t^{(r)^2} = -\mathbf{I}^{(r)} \qquad (t = 1, 2, \ldots, r),$$

$$\mathbf{K}_t^{(r)}\mathbf{K}_{t'}^{(r)} + \mathbf{K}_{t'}^{(r)}\mathbf{K}_t^{(r)} = \mathbf{O}^{(r)} \qquad (t = 1, 2, \ldots, r-1; t' = t+1, t+2, \ldots, r). \tag{25}$$

Proof. The proof follows by induction. We easily verify that

$$\mathbf{K}_1^{(1)^2} = -\mathbf{I}^{(1)} \tag{26}$$

and also that

$$\mathbf{K}_1^{(1)}\tilde{\mathbf{I}}^{(1)} = -\tilde{\mathbf{I}}^{(1)}\mathbf{K}_1^{(1)}. \tag{27}$$

Now assume that Eqs. (24) are true with r replaced by $r-1$, and in addition that

$$\mathbf{K}_t^{(r-1)}\tilde{\mathbf{I}}^{(r-1)} = -\tilde{\mathbf{I}}^{(r-1)}\mathbf{K}_t^{(r-1)} \qquad (t = 1, 2, \ldots, r-1).$$

We then have

$$\mathbf{K}_t^{(r)^2} = \begin{pmatrix} \mathbf{K}_t^{(r-1)^2} & \mathbf{O}^{(r-1)} \\ \mathbf{O}^{(r-1)} & \mathbf{K}_t^{(r-1)^2} \end{pmatrix} \qquad (t = 1, 2, \ldots, r-1),$$

and also, trivially,

$$\mathbf{K}_r^{(r)^2} = -\mathbf{I}^{(r)}.$$

Furthermore

$$\mathbf{K}_t^{(r)}\mathbf{K}_{t'}^{(r)} + \mathbf{K}_{t'}^{(r)}\mathbf{K}_t^{(r)}$$

$$= \begin{pmatrix} \mathbf{K}_t^{(r-1)}\mathbf{K}_{t'}^{(r-1)} + \mathbf{K}_{t'}^{(r-1)}\mathbf{K}_t^{(r-1)} & \mathbf{O}^{(r-1)} \\ \mathbf{O}^{(r-1)} & \mathbf{K}_t^{(r-1)}\mathbf{K}_{t'}^{(r-1)} + \mathbf{K}_{t'}^{(r-1)}\mathbf{K}_t^{(r-1)} \end{pmatrix}$$

$$(t = 1, 2, \ldots, r-1; t' = t+1, t+2, \ldots, r-1),$$

and in addition

$$\mathbf{K}_r^{(r)}\mathbf{K}_t^{(r)} + \mathbf{K}_t^{(r)}\mathbf{K}_r^{(r)}$$

$$= \begin{pmatrix} \mathbf{0}^{(r-1)} & \mathbf{K}_t^{(r-1)}\tilde{\mathbf{I}}^{(r-1)} + \tilde{\mathbf{I}}^{(r-1)}\mathbf{K}_t^{(r-1)} \\ -\mathbf{K}_t^{(r-1)}\tilde{\mathbf{I}}^{(r-1)} - \tilde{\mathbf{I}}^{(r-1)}\mathbf{K}_t^{(r-1)} & \mathbf{0}^{(r-1)} \end{pmatrix}$$

$$(t = 1, 2, \ldots, r-1).$$

Finally

$$\mathbf{K}_t^{(r)}\tilde{\mathbf{I}}^{(r)} = \begin{pmatrix} \mathbf{0}^{(r-1)} & \mathbf{K}_t^{(r-1)}\ \tilde{\mathbf{I}}^{(r-1)} \\ \mathbf{K}_t^{(r-1)}\ \tilde{\mathbf{I}}^{(r-1)} & \mathbf{0}^{(r-1)} \end{pmatrix} \qquad (t = 1, 2, \ldots, r-1),$$

$$\mathbf{K}_r^{(r)}\tilde{\mathbf{I}}^{(r)} = \begin{pmatrix} \mathbf{I}^{(r-1)} & \mathbf{0}^{(r-1)} \\ \mathbf{0}^{(r-1)} & -\mathbf{I}^{(r-1)} \end{pmatrix};$$

$$\tilde{\mathbf{I}}^{(r)}\mathbf{K}_t^{(r)} = \begin{pmatrix} \mathbf{0} & \tilde{\mathbf{I}}^{(r-1)}\mathbf{K}_t^{(r-1)} \\ \tilde{\mathbf{I}}^{(r-1)}\mathbf{K}_t^{(r-1)} & \mathbf{0} \end{pmatrix} \qquad (t = 1, 2, \ldots, r-1),$$

$$\tilde{\mathbf{I}}^{(r)}K_r^{(r)} = \begin{pmatrix} -\mathbf{I}^{(r-1)} & \mathbf{0}^{(r-1)} \\ \mathbf{0}^{(r-1)} & \mathbf{I}^{(r-1)} \end{pmatrix}.$$

Thus if formulas (25) are correct for the set of matrices with superscript $r - 1$, they are also true for the set with superscript r; it follows immediately from Eqs. (26) and (27) that they are generally true.

It is now a simple matter to construct sets of matrices satisfying Eqs. (23); two such are given in

THEOREM 2. *The matrices*

$$\mathbf{\Gamma}_t^{(n)} = i\mathbf{K}_t^{(n')} \qquad (t = 1, 2, \ldots, n) \tag{28}$$

and

$$\mathbf{\Gamma}_t^{(n)} = \mathbf{K}_1^{(n'+1)}\mathbf{K}_{t+1}^{(n'+1)} \qquad (t = 1, 2, \ldots, n), \tag{29}$$

where $n' \geqslant n$ *in both cases, satisfy Eqs.* (23).

The proof of this theorem is trivial. The matrices of Eqs. (28) are purely imaginary, those of Eqs. (29) are purely real; the matrices of the first set are perhaps the simpler, and we make use of them in the sequel.

10. A COMPLEX NUMBER EXTENSION TO THE CLIFFORD ALGEBRA [48]

We now turn to the first of the difficulties mentioned earlier; we must devise an isomorphism between vectors and matrices which adumbrates the inversion of vectors with complex components.

In order to distinguish the real and imaginary parts of the components of the given vector $\mathbf{z}$, we let $\mathbf{z} \leftrightarrow \mathbf{Z}$ with

$$\mathbf{Z} = \sum_{t=1}^{n} x_t \mathbf{X}_t + \sum_{t=1}^{n} y_t \mathbf{Y}_t, \tag{30}$$

where in the notation of Eq. (2)

$$z_t = x_t + i y_t \qquad (t = 1, 2, \ldots, n).$$

If

$$\tilde{\mathbf{Z}} = \sum_{t=1}^{n} x_t \mathbf{X}_t - \sum_{t=1}^{n} y_t \mathbf{Y}_t$$

is the matrix isomorphic to the complex conjugate vector, then, from the definition of a vector inverse,

$$\mathbf{z}^{-1} \leftrightarrow \left\{ \sum_{t=1}^{n} (x_t^2 + y_t^2) \right\}^{-1} \tilde{\mathbf{Z}},$$

and if the isomorphism between vectors and matrices is to be preserved during inversion, we must have

$$\mathbf{Z}^{-1} = \left\{ \sum_{t=1}^{n} (x_t^2 + y_t^2) \right\}^{-1} \tilde{\mathbf{Z}}, \tag{31}$$

or

$$\mathbf{Z}\tilde{\mathbf{Z}} = \left\{ \sum_{t=1}^{n} (x_t^2 + y_t^2) \right\} \mathbf{I}. \tag{32}$$

Actual multiplication yields

$$\begin{aligned} \mathbf{Z}\tilde{\mathbf{Z}} = & \sum_{t=1}^{n} x_t^2 \mathbf{X}_t^2 + \sum_{t=1}^{n-1} \sum_{t'=t+1}^{n} x_t x_{t'} (\mathbf{X}_t \mathbf{X}_{t'} + \mathbf{X}_{t'} \mathbf{X}_t) \\ & - \sum_{t=1}^{n} \sum_{t'=1}^{n} x_t y_{t'} \mathbf{X}_t \mathbf{Y}_{t'} + \sum_{t=1}^{n} \sum_{t'=1}^{n} x_t y_{t'} \mathbf{Y}_{t'} \mathbf{X}_t \\ & - \sum_{t=1}^{n} y_t^2 \mathbf{Y}_t^2 - \sum_{t=1}^{n-1} \sum_{t'=t+1}^{n} y_t y_{t'} (\mathbf{Y}_t \mathbf{Y}_{t'} + \mathbf{Y}_{t'} \mathbf{Y}_t). \end{aligned} \tag{33}$$

If Eqs. (32) and (33) are to be equivalent, we must have identically

$$
\begin{aligned}
\mathbf{X}_t^2 &= \mathbf{I} && (t = 1, 2, \ldots, n),\\
\mathbf{X}_t\mathbf{X}_{t'} + \mathbf{X}_{t'}\mathbf{X}_t &= \mathbf{0} && (t = 1, 2, \ldots, n-1; t' = t+1, t+2, \ldots, n),\\
\mathbf{X}_t\mathbf{Y}_{t'} &= \mathbf{Y}_{t'}\mathbf{X}_t && (t = 1, 2, \ldots, n; t' = 1, 2, \ldots, n),\\
\mathbf{Y}_t^2 &= -\mathbf{I} && (t = 1, 2, \ldots, n),\\
\mathbf{Y}_t\mathbf{Y}_{t'} + \mathbf{Y}_{t'}\mathbf{Y}_t &= \mathbf{0} && (t = 1, 2, \ldots, n-1; t' = t+1, t+2, \ldots, n).
\end{aligned}
\tag{34}
$$

There are many ways of constructing the required set of matrices; perhaps the simplest is described in

THEOREM 3. *If* $\mathbf{\Gamma}_t^{(n')}$ $(t = 1, 2, \ldots, n'; n' \geqslant 2n+1)$ *constitutes a set of matrices obeying Eqs.* (23), *then the set*

$$
\left.
\begin{aligned}
\mathbf{X}_t &= \mathbf{\Gamma}_{2t}^{(n')}\\
\mathbf{Y}_t &= \mathbf{\Gamma}_1^{(n')}\mathbf{\Gamma}_{2t+1}^{(n')}
\end{aligned}
\right\}
\quad (t = 1, 2, \ldots, n)
\tag{35}
$$

obeys Eqs. (34).

11. THE VECTOR-VALUED PADÉ TABLE

In the further development of the theory of vector continued fractions, we are particularly anxious to ensure that vector inversion is a reflexive process; we wish to consider vectors $\mathbf{z}$ of such a class, that with the single exception of the zero vector

$$(\mathbf{z}^{-1})^{-1} = \mathbf{z}.$$

Clearly all finite vectors of finite dimension belong to the required class. However, if the vector

$$\mathbf{z} = (z_1, z_2, \ldots)$$

has an infinite number of components, then we must have

$$\sum_{t=1}^{\infty} z_t \bar{z}_t < \infty; \tag{36}$$

we restrict our attention, that is to say, to L_2 vectors.

We must now consider matrices $\mathbf{Z}$, given by an equation of the form (30), which are of infinite dimension, either because the corresponding vector $\mathbf{z}$ is of infinite dimension or because the generator matrices $\mathbf{X}_t$, $\mathbf{Y}_t$ $(t = 1, 2, \ldots, n)$ have been chosen to be of infinite dimension. In a general context the inversion of infinite matrices is beset by questions of uniqueness. However, in our case the matrices $\mathbf{Z}$ which we consider are so sparse that such questions hardly arise; it is easy to verify that subject to condition (35), $\mathbf{Z}^{-1}$ (as defined by Eq. (31) with n finite or infinite depending on the nature of $\mathbf{z}$) is both a left-hand and a right-hand inverse of $\mathbf{Z}$; $\mathbf{Z}^{-1}$ is thus uniquely determined not only by Eq. (30) but also in the sense of matrix inversion.

We wish now to remark that the inverse of a nonzero L_2 vector is also an L_2 vector and, as is well known, the sum and difference of two such vectors are also L_2 vectors. The vector ε algorithm employs vector addition, subtraction, and inversion; thus if one excludes the possibility that one is called upon to invert a vector which is identically zero, it follows that if the initial vectors from which the ε array is built up are all L_2 vectors, then the ε array itself is composed of such vectors.

We recapitulate the results of the preceding paragraphs in

THEOREM 4. *The sequence* $\{\mathbf{u}_t\}$ *of complex-valued* L_2 *vectors is given. The* ε *array of vector-valued rational functions* $\{\varepsilon_r^{(m)}\}$ $(r = 0, 1, \ldots;\ m = -(r \div 2) - 1, -(r \div 2), \ldots)$ *can formally be constructed from the initial values*

$$\varepsilon_{-1}^{(m)} = 0 \quad (m = 0, 1, \ldots), \qquad \varepsilon_{2m}^{(-m-1)} = 0 \quad (m = 0, 1, \ldots),$$

$$\varepsilon_0^{(m)} = \sum_{t=0}^{m} \mathbf{u}_t \mu^t \qquad (m = 0, 1, \ldots),$$

by use of the relationships (4) *with* $r = 0, 1, \ldots;\ m = -(r \div 2) - 1, -(r \div 2), \ldots$. *The only reason for which the formation of the* ε *array can break down is that fortuitously one or more vertical pairs of* ε *vectors (for example,* $\boldsymbol{\epsilon}_r^{(m+1)}$ *and* $\boldsymbol{\epsilon}_r^{(m)}$*) are equal. For those sectors of the vector* ε *array which can be constructed, the resulting vectors* $\{\boldsymbol{\epsilon}_r^{(m)}\}$ *are* L_2 *vectors.*

The coefficients $\{\mathbf{u}_t\}$ *may be translated into matrix equivalents*

$$\mathbf{U}_t \leftrightarrow \mathbf{u}_t \qquad (t = 0, 1, \ldots) \tag{37}$$

according to an isomorphism of the form (29); *the* ε *array of matrix-valued rational functions* $\mathbf{E}_r^{(m)}$ $(r = 0, 1, \ldots;\ m = -(r \div 2) - 1, -(r \div 2), \ldots)$

can formally be constructed from the initial values (21) *by use of the relationships* (16) *with* $r = 0, 1, \ldots; m = -(r \div 2) - 1, -(r \div 2), \ldots$. *Those sectors of the matrix* ε *array which can be constructed correspond completely to those sectors of the vector* ε *array which can be constructed; furthermore in these sectors and according to the selected isomorphism*

$$\boldsymbol{\epsilon}_r^{(m)} \leftrightarrow \mathbf{E}_r^{(m)}. \tag{38}$$

Again the equations of this theorem may be regarded in two lights. First, they may be considered to represent relationships between vector- and matrix-valued rational functions in which μ occurs as a variable; second, the value of μ may be considered to be fixed: Eqs. (4) and (16) then concern the numerical values of vectors and matrices respectively.

Theorem 4 may be given an alternative interpretation: the sequence $\{\mathbf{s}_m\}$ of complex-valued L_2 vectors is given. The ε array of complex-valued vectors $\{\boldsymbol{\epsilon}_r^{(m)}\}$ can formally be constructed from the initial values (1) by use of relationships (4) with $r = 0, 1, \ldots; m = 0, 1, \ldots$. Again the only reason for breakdown of the ε algorithm is the presence of equal vertical pairs of ε vectors. In those sectors of the ε array which can be constructed the resulting vectors $\{\boldsymbol{\epsilon}_r^{(m)}\}$ belong to the class L_2. The vectors $\{\mathbf{s}_m\}$ may be translated into matrix equivalents $\{\mathbf{S}_m\}$ according to an isomorphism of the form (30). The ε array of matrices $\{\mathbf{E}_r^{(m)}\}$ can formally be constructed from the initial values (22) by use of the relationships (16) again with $r = 0, 1, \ldots; m = 0, 1, \ldots$. Again the sector of the vector and matrix ε arrays which can be constructed correspond, and in these sectors Eq. (38) holds.

It is this interpretation which finally absolves the vector ε algorithm from any suspicion of being a mere adventitious computational trick, and offers a perfectly secure interpretation of the vector ε array. We may regard any given sequence $\{\mathbf{s}_m\}$ as being the successive partial sums of a vector series according to the formula

$$s_m = \sum_{t=0}^{m} \mathbf{u}_t \mu^t,$$

where μ is any fixed, finite, nonzero real number. The even order transformed vectors $\{\boldsymbol{\epsilon}_{2r}^{(m)}\}$ are isomorphic to the matrix-valued Padé quotients derived from the series $\sum_{t=0}^{\infty} \mathbf{U}_t \mu^t$, where Eq. (37) holds. The convergence properties of the vectors $\{\boldsymbol{\epsilon}_{2r}^{(m)}\}$ may be deduced from the convergence theory of operator-valued continued fractions. We may speak of the even order vector ε array as being a vector-valued Padé table.

12. G NUMBERS

Clifford numbers are not, in general, elements of a binary algebra: although both $\mathbf{Z}$ and $\mathbf{Z}'$ may be Clifford numbers, the product $\mathbf{Z}\mathbf{Z}'$ is not. Nevertheless Clifford numbers possess certain interesting formal properties; for example, it may be shown that such combinations as $\mathbf{Z}\mathbf{Z}'\mathbf{Z}$ and $\mathbf{Z}\mathbf{Z}'\mathbf{Z}'' + \mathbf{Z}''\mathbf{Z}'\mathbf{Z}$ are in fact Clifford numbers. We are able to extend these results to our representations of vectors with complex coefficients.

In order to present the inquiry in the most general terms, we introduce

DEFINITION 1. *A number of the form*

$$\mathbf{G} = \sum_{t=1}^{n} x_t \mathbf{X}_t + \sum_{t=1}^{n'} y_t \mathbf{Y}_t, \tag{39}$$

where the numbers $\{\mathbf{X}_t\}$ *and* $\{\mathbf{Y}_t\}$ *satisfy the equations*

$$\begin{aligned}
\mathbf{X}_t^2 &= \mathbf{I} && (t = 1, 2, \ldots, n),\\
\mathbf{X}_t\mathbf{X}_{t'} + \mathbf{X}_{t'}\mathbf{X}_t &= \mathbf{0} && (t = 1, 2, \ldots, n-1; t' = t+1, t+2, \ldots, n),\\
\mathbf{X}_t\mathbf{Y}_{t'} &= \mathbf{Y}_{t'}\mathbf{X}_t && (t = 1, 2, \ldots, n; t' = 1, 2, \ldots, n'),\\
\mathbf{Y}_t^2 &= -\mathbf{I} && (t = 1, 2, \ldots, n'),\\
\mathbf{Y}_t\mathbf{Y}_{t'} + \mathbf{Y}_{t'}\mathbf{Y}_t &= \mathbf{0} && (t = 1, 2, \ldots, n'-1; t' = t+1, t+2, \ldots, n')
\end{aligned} \tag{40}$$

and the coefficients $\{x_t\}$ *and* $\{y_t\}$ *are real or complex numbers for which*

$$\sum_{t=1}^{n} |x_t|^2 + \sum_{t=1}^{n'} |y_t|^2 < \infty, \tag{41}$$

is called a G number; we write

$$((\mathbf{G})) = \sum_{t=1}^{n} x_t^2 + \sum_{t=1}^{n'} y_t^2;$$

if the numbers $\mathbf{G}, \mathbf{G}', \mathbf{G}'', \ldots$ *all have representations of the form* (39) *subject to the conditions* (40) *in which the numbers* $\{\mathbf{X}_t\}$, $\{\mathbf{Y}_t\}$ *are the same, then we write* $\mathbf{G}, \mathbf{G}', \mathbf{G}'', \ldots \in \mathfrak{G}$.

The matrices $\mathbf{Z}$ of Eq. (30) are special cases of G numbers. We denote the number conjugate to $\mathbf{G}$ by

$$\tilde{\mathfrak{G}} = \sum_{t=1}^{n} x_t \mathbf{X}_t - \sum_{t=1}^{n'} y_t \mathbf{Y}_t.$$

Clearly $\mathfrak{G}$, $\tilde{\mathfrak{G}} \in \mathfrak{G}$.

The results of this section are based on the fundamental

THEOREM 5. *If* $\mathfrak{G}, \mathfrak{G}' \in \mathfrak{G}$ *with coefficients* $\{x_t\}$, $\{y_t\}$ *and* $\{x_t'\}$, $\{y_t'\}$, *respectively, then*

$$\{\mathfrak{G}, \mathfrak{G}'\} = \mathfrak{G}\tilde{\mathfrak{G}}' + \mathfrak{G}'\tilde{\mathfrak{G}}, \tag{42}$$

$$= 2\left(\sum_{t=1}^{n} x_t x_{t'} + \sum_{t=1}^{n'} y_t y_t'\right)\mathbf{I}. \tag{43}$$

Proof. This result is easily verified by direct substitution of expressions of the form (39), together with their conjugate expressions, into formula (42), and use of Eqs. (40).

DEFINITION 2. *If* $\mathfrak{G}, \mathfrak{G}' \in \mathfrak{G}$, *and in the notation of Eq.* (43)

$$\{\mathfrak{G}, \mathfrak{G}'\} = 0,$$

then we write $\mathfrak{G} \perp \mathfrak{G}'$.

Clearly perpendicularity as we have defined it is reflexive: if $\mathfrak{G} \perp \mathfrak{G}'$, then $\mathfrak{G}' \perp \mathfrak{G}$.

We can now proceed to

THEOREM 6. *If* $\mathfrak{G}, \mathfrak{G}' \in \mathfrak{G}$, *then* $\mathfrak{G}\mathfrak{G}'\mathfrak{G} \in \mathfrak{G}$; *specifically*

$$\mathfrak{G}\mathfrak{G}'\mathfrak{G} = \{\mathfrak{G}, \tilde{\mathfrak{G}}'\}\mathfrak{G} - ((\mathfrak{G}))\tilde{\mathfrak{G}}'. \tag{44}$$

Proof. We write

$$\mathfrak{G}\mathfrak{G}'\mathfrak{G} = \mathfrak{G}(\mathfrak{G}'\mathfrak{G} + \tilde{\mathfrak{G}}\tilde{\mathfrak{G}}') - \mathfrak{G}\tilde{\mathfrak{G}}\tilde{\mathfrak{G}}'$$

and Eq. (44) follows immediately.

COROLLARY. *If* $\tilde{\mathfrak{G}}' \perp \mathfrak{G}$ *then*

$$\mathfrak{G}\mathfrak{G}'\mathfrak{G} = -((\mathfrak{G}))\tilde{\mathfrak{G}}'.$$

THEOREM 7. *If* $\mathbf{G}, \mathbf{G}', \mathbf{G}'' \in \mathfrak{G}$, *then* $\mathbf{G}\mathbf{G}'\mathbf{G}'' + \mathbf{G}''\mathbf{G}'\mathbf{G} \in \mathfrak{G}$; *specifically*

$$\mathbf{G}\mathbf{G}'\mathbf{G}'' + \mathbf{G}''\mathbf{G}'\mathbf{G} = \{\mathbf{G}', \tilde{\mathbf{G}}''\}\mathbf{G} + \{\mathbf{G}', \tilde{\mathbf{G}}\}\mathbf{G}'' - \{\mathbf{G}, \mathbf{G}''\}\tilde{\mathbf{G}}'. \tag{45}$$

Proof. We write

$$\mathbf{G}\mathbf{G}'\mathbf{G}'' = \mathbf{G}(\mathbf{G}'\mathbf{G}'' + \tilde{\mathbf{G}}''\tilde{\mathbf{G}}') - \mathbf{G}\tilde{\mathbf{G}}''\tilde{\mathbf{G}}',$$

$$\mathbf{G}''\mathbf{G}'\mathbf{G} = \mathbf{G}''(\mathbf{G}'\mathbf{G} + \tilde{\mathbf{G}}\tilde{\mathbf{G}}') - \mathbf{G}''\tilde{\mathbf{G}}\tilde{\mathbf{G}}'$$

and have

$$\mathbf{G}\mathbf{G}'\mathbf{G}'' + \mathbf{G}''\mathbf{G}'\mathbf{G} = \mathbf{G}(\mathbf{G}'\mathbf{G}'' + \tilde{\mathbf{G}}''\tilde{\mathbf{G}}') + \mathbf{G}''(\mathbf{G}'\mathbf{G} + \tilde{\mathbf{G}}\tilde{\mathbf{G}}')$$
$$- (\mathbf{G}\tilde{\mathbf{G}}'' + \mathbf{G}''\tilde{\mathbf{G}})\tilde{\mathbf{G}}',$$

which reduces to Eq. (45).

COROLLARY. *If* $\tilde{\mathbf{G}} \perp \mathbf{G}'$, $\mathbf{G} \perp \mathbf{G}''$, $\mathbf{G}' \perp \tilde{\mathbf{G}}''$, *then*

$$\mathbf{G}\mathbf{G}'\mathbf{G}'' + \mathbf{G}''\mathbf{G}'\mathbf{G} = \mathbf{0}.$$

There are a number of indications that the elementary results concerning G numbers which we have just derived have fundamental application to the theory underlying the vector-valued Padé table.

For example, we may easily show that

$$pre\left[\frac{\mathbf{U}_m}{\lambda - \mathbf{U}_{m+1}\mathbf{U}_m^{-1}}\right] = post\left[\frac{\mathbf{U}_m}{\lambda - \mathbf{U}_m^{-1}\mathbf{U}_{m+1}}\right]$$
$$= (\mathbf{U}_m^{-1}\lambda - \mathbf{U}_m^{-1}\mathbf{U}_{m+1}\mathbf{U}_m^{-1})^{-1}.$$

If $\mathbf{U}_m$ and $\mathbf{U}_{m+1}$ are G numbers, then so are $\mathbf{U}_m^{-1}$ and (from Theorem 6) $\mathbf{U}_m^{-1}\mathbf{U}_{m+1}\mathbf{U}_m^{-1}$. Thus we have been able to express the first convergent of the G number continued fraction associated with a G number coefficient power series solely in terms of G numbers or, taking a special case, solely in terms of vectors.

Although the convergents of the associated continued fraction derived from a G-number coefficient power series are of course G numbers, their coefficients are not. It is an interesting speculation and would be a substantial achievement in the theory of vector continued fractions if the auxiliary theory of operator continued fractions derived from power series could be so versed as to consist solely of such expressions as are encountered in Theorems 6 and 7; when this theory is restricted to power series whose coefficients are matrices isomorphic to vectors it would then transpire that the formalism of the vector-valued Padé table can entirely be expressed in terms of vectors, and not in terms of a mixture of matrices and vectors as is the case at the moment.

We conclude this section by showing that in general a G number satisfies a quartic equation with scalar coefficients.

THEOREM 8. *Let* $\mathbf{G}$ *be a* G *number given by Eq.* (39), *and write*

$$x^2 = \sum_{t=1}^{n} x_t^2, \qquad y^2 = \sum_{t=1}^{n'} y_t^2. \tag{46}$$

(1) *If* $n = 0$, $\mathbf{G}$ *satisfies the quadratic equation*

$$\mathbf{G}^2 + y^2\mathbf{I} = \mathbf{0}; \tag{47}$$

(2) *If* $n' = 0$, $\mathbf{G}$ *satisfies*

$$\mathbf{G}^2 - x^2\mathbf{I} = \mathbf{0}; \tag{48}$$

(3) *If* $nn' \neq 0$, *then* $\mathbf{G}$ *satisfies the quartic equation*

$$\mathbf{G}^4 - 2(x^2 - y^2)\mathbf{G}^2 + (x^2 + y^2)^2\mathbf{I} = \mathbf{0}. \tag{49}$$

Proof. The proofs of the first two parts of the theorem are trivial. To prove the third, we write

$$\mathbf{X} = \sum_{t=1}^{n} x_t\mathbf{X}_t, \qquad \mathbf{Y} = \sum_{t=1}^{n} y_t\mathbf{Y}_t,$$

so that

$$\mathbf{G} = \mathbf{X} + \mathbf{Y}$$

$$\mathbf{X}^2 = x^2\mathbf{I}, \qquad \mathbf{Y}^2 = -y^2\mathbf{I},$$

and

$$\mathbf{XY} = \mathbf{YX}.$$

We then have

$$\mathbf{G}^2 = (x^2 - y^2)\mathbf{I} + 2\mathbf{XY}$$

$$\mathbf{G}^4 = \{(x^2 - y^2)^2 - 4x^2y^2\}\mathbf{I} + 4(x^2 - y^2)\mathbf{XY},$$

and by eliminating the number $\mathbf{XY}$, we obtain Eq. (49).

13. H MATRICES AND SOME RESULTS CONCERNING NORMS

In this section we consider a specific representation of G numbers.

DEFINITION 3. *An H matrix has the form*

$$\mathbf{H} = \sum_{t=1}^{n} x_t \mathbf{X}_t + \sum_{t=1}^{n'} y_t \mathbf{Y}_t, \tag{50}$$

where

$$\begin{aligned} \mathbf{X}_t &= \mathbf{\Gamma}_{\eta_t}^{(r)} \qquad (t = 1, 2, \ldots, n) \\ \mathbf{Y}_t &= \mathbf{\Gamma}_1^{(r)} \mathbf{\Gamma}_{\eta_t'}^{(r)} \qquad (t = 1, 2, \ldots, n'), \end{aligned} \tag{51}$$

where the numbers $\eta_1, \eta_1', \eta_2, \eta_2', \ldots$ make up the integer set $(2, 3, \ldots, n + n' + 1)$, $r \geqslant n + n' + 1$, $\mathbf{\Gamma}_t^{(r)}$ $(t = 1, 2, \ldots, n + n' + 1)$ are matrices given by Eqs. (28); *$\{x_t\}$, $\{y_t\}$ are real or complex numbers satisfying relationship* (41).

Concerning the spectrum of an H matrix, we have

THEOREM 9. *x and y have the meanings attached to them in Eqs.* (46).

(a) *If, in the notation of Eq.* (51), *$r < \infty$, then if either $n = 0$ or $n' = 0$, $\mathbf{H}$ has two eigenvalues (in the first case $\pm iy$, in the second $\pm x$) each of multiplicity 2^{r-1}; if $nn' \neq 0$, $\mathbf{H}$ has four eigenvalues $\pm x \pm iy$, each of multiplicity 2^{r-2}.*

(b) *If $r = \infty$ then the infinite matrix $\mathbf{H}$ has a point spectrum. If either $n = 0$ or $n' = 0$, then there are two such points, in the first case $\pm iy$ and in the second $\pm x$; if $nn' \neq 0$, then there are four such points $\pm x \pm iy$.*

Proof. It follows from Eqs. (47–49) that the spectrum of $\mathbf{H}$ is limited to the points mentioned. Concerning the multiplicities of the eigenvalues, we remark first that each eigenvalue of $\mathbf{H}$ is a continuous function of each member of the sets $\{x_t\}$, $\{y_t\}$, and second that it is easy to verify that for finite r, the matrix $\mathbf{K}_t^{(r)}$ has two eigenvalues $\pm i$ each of multiplicity 2^{r-1}, and that for infinite r, $\mathbf{K}_t^{(r)}$ has a two-point spectrum $\pm i$. The theorem then follows.

We now derive a result concerning the structure of H matrices:

THEOREM 10. *Accompanied of a factor of $\pm i$ or ± 1, each element of the sets $\{x_t\}$, $\{y_t\}$ occurs once and only once in each row and in each column of the H matrix* (50).

Proof. We examine the matrices $\{\mathbf{K}_t^{(r)}\}$ occurring in Eq. (28).

For a fixed t the matrix $\mathbf{K}_t^{(r)}$ $(1 \leqslant t \leqslant r)$ has one and only one nonzero element in every row and also in every column; moreover, if $\mathbf{K}_t^{(r)}$ has a nonzero element in a certain position, then the elements in this position in all other matrices belonging to the set $\{\mathbf{K}_t^{(r)}\}$ (r being fixed) are all zero. Furthermore premultiplication of a suitable matrix by $\mathbf{\Gamma}_1^{(r)}$ is equivalent to an interchange of the first and second rows, the third and fourth rows, and so on, together with the scalar multiplication of certain of these rows by $+i$, and of the others by $-i$. The result contained in the theorem then follows by considering the linear combination (50).

Our third result in this section concerns the conjugate matrix $\tilde{\mathbf{H}}$.

THEOREM 11. *If the coefficients* $\{x_t\}$ *and* $\{y_t\}$ *are real numbers, then*

$$\tilde{\mathbf{H}} = \mathbf{H}^*,$$

where the asterisk denotes a complex conjugate transpose.

Proof. In the representation of Eq. (50), the matrices $\{\mathbf{X}_t\}$ are purely imaginary and the $\{\mathbf{Y}_t\}$ are purely real. Transposing any matrix $\mathbf{X}_t$ or $\mathbf{Y}_t$ defined by Eqs. (51) about its principal diagonal is equivalent to multiplication throughout by -1. Replacing i by $-i$ changes the sign of each element of $\mathbf{X}_t$ and leaves the elements of $\mathbf{Y}_t$ unaltered. The result of the theorem follows immediately.

We conclude by deriving some results concerning the norms of H matrices.

It will be recalled that in practice use is made of three principal vector norms (see, for example, [49]): if

$$\mathbf{v} = (v_1, v_2, \ldots, v_m)$$

is a typical vector with complex components, the norms in question are defined by

$$||\mathbf{v}||_1 = \sum_{t=1}^{m} |v_t|, \tag{52}$$

$$||\mathbf{v}||_2 = \left\{\sum_{t=1}^{m} |v_t|^2\right\}^{1/2}, \tag{53}$$

$$||\mathbf{v}||_\infty = \max_{t=1,2,\ldots,m} |v_t|.$$

If

$$\mathbf{A} = (a_{i,j})$$

is a typical $m \times m$ matrix of complex numbers, then three matrix norms which are compatible with these vector norms, in the sense that $||\mathbf{Av}|| \leqslant ||\mathbf{A}||\ ||\mathbf{v}||$, are

$$||\mathbf{A}||_1 = \max_j \sum_{i=1}^{m} |a_{i,j}|, \tag{54}$$

$$||\mathbf{A}||_2 = (\text{maximum eigenvalue of } \mathbf{AA}^*)^{1/2}, \tag{55}$$

$$||\mathbf{A}||_\infty = \max_i \sum_{j=1}^{m} |a_{i,j}|, \tag{56}$$

respectively. A fourth matrix norm, the Euclidean or Schur norm, defined by

$$||\mathbf{A}||_E = \left(\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{i,j}|^2 \right)^{1/2}, \tag{57}$$

may be used to replace the $||\mathbf{A}||_2$ norm in the above scheme, but its numerical value can in fact be larger than that of $||\mathbf{A}||_2$ by a factor $m^{1/2}$; indeed we have

$$||\mathbf{A}||_2 \leqslant ||\mathbf{A}||_E \leqslant m^{1/2} ||\mathbf{A}||_2. \tag{58}$$

Our study of the structure and properties of H matrices has been sufficiently extensive to enable us to give a comprehensive account of their norms:

THEOREM 12. *In the notation of Eqs.* (50) *and* (54)–(58)

$$||\mathbf{H}||_1 = ||\mathbf{H}||_\infty = \sum_{t=1}^{n} |x_t| + \sum_{t=1}^{n} |y_t|,$$

(if r is infinite, i.e., $\mathbf{H}$ *is an infinite matrix, then unless all coefficients $\{x_t\}$, $\{y_t\}$ are zero, $||\mathbf{H}||_E$ is necessarily infinite). If we introduce the additional restriction that the coefficients $\{x_t\}$, $\{y_t\}$ should be real, then in the notation of Eqs.* (46)

$$||\mathbf{H}||_2 = (x^2 + y^2)^{1/2};$$

furthermore, if r is finite, then

$$||\mathbf{H}||_E = m^{1/2}||\mathbf{H}||_2,$$

where $m = 2^{r+1}$.

Proof. The above results can readily be verified by an appeal to Theorems 10 and 11, and use of Eqs. (54)–(58).

We conclude by recording some curious relationships between the norms of vectors and those of their companion matrices as given by the isomorphism of Eq. (30).

THEOREM 13. *If the vector* $\mathbf{z}$ *and its companion matrix* $\mathbf{Z}$ *are given by Eqs.* (2) *and the set* (30), (35), (28) *respectively, then*

$$||\mathbf{z}||_1 = ||\mathbf{Z}||_1 = ||\mathbf{Z}||_\infty,$$

$$||\mathbf{z}||_2 = ||\mathbf{Z}||_2.$$

Proof. The above results follow from Eqs. (52) and (53), and use of the previous theorem.

The significance of the last result is that there is an isomorphism not only between the formalisms of matrix and vector continued fractions, but also between their convergence theories.

APPENDIX A. HYPERQUATERNION VECTORS

In this appendix we are concerned with a matter which must be dismissed at some stage, although it does not appear to contribute essentially to the development of the theory.

It will be recalled that the inverse of the quaternion

$$q = x + y^{(1)}\mathbf{i} + y^{(2)}\mathbf{j} + y^{(3)}\mathbf{k}$$

is given by

$$q^{-1} = (x^2 + y^{(1)^2} + y^{(2)^2} + y^{(3)^2})^{-1}\bar{q},$$

where

$$\bar{q} = x - y^{(1)}\mathbf{i} - y^{(2)}\mathbf{j} - y^{(3)}\mathbf{k}.$$

This immediately suggests that the concept of the inverse of a vector whose components are complex numbers may be extended to quaternion vectors: given the vector

$$\mathfrak{q} = (q_1, q_2, \ldots, q_n),$$

where

$$q_t = x_t + y_t^{(1)}\mathbf{i} + y_t^{(2)}\mathbf{j} + y_t^{(3)}\mathbf{k} \qquad (t = 1, 2, \ldots, n),$$

then

$$\mathfrak{q}^{-1} = \left\{\sum_{t=1}^{n}\left(x_t^2 + \sum_{m=1}^{3} y_t^{(m)^2}\right)\right\}^{-1}\bar{\mathfrak{q}},$$

where

$$\bar{\mathfrak{q}} = (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n).$$

Again the extension may be continued: we suppose that associated with each element of a vector there are κ timelike coordinates and κ' spacelike coordinates; thus we now consider the vector

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n), \tag{59}$$

where

$$\omega_t = \sum_{l=1}^{\kappa'} x_t^{(l)}\boldsymbol{\sigma}_l + \sum_{m=1}^{\kappa} y_t^{(m)}\boldsymbol{\tau}_m \qquad (t = 1, 2, \ldots, n),$$

and define the inverse of (59) to be

$$\boldsymbol{\omega}^{-1} = \left\{\sum_{t=1}^{n}\left(\sum_{l=1}^{\kappa'} x_t^{(l)^2} + \sum_{m=1}^{\kappa} y_t^{(m)^2}\right)\right\}^{-1}\bar{\boldsymbol{\omega}}, \tag{60}$$

where

$$\bar{\boldsymbol{\omega}} = (\bar{\omega}_1, \bar{\omega}_2, \ldots, \bar{\omega}_n),$$

$$\bar{\omega}_t = \sum_{l=1}^{\kappa'} x_t^{(l)}\boldsymbol{\sigma}_l - \sum_{m=1}^{\kappa} y_t^{(m)}\boldsymbol{\tau}_m \qquad (t = 1, 2, \ldots, n).$$

We can easily construct an isomorphism which preserves addition, subtraction, and inversion according to formula (60), between such vectors and certain matrices. We put

$$\mathbf{\Omega} = \sum_{t=1}^{n}\left(\sum_{l=1}^{\kappa'} x_t^{(l)}\mathbf{X}_t^{(l)} + \sum_{m=1}^{\kappa} y_t^{(m)}\mathbf{Y}_t^{(m)}\right),$$

and find that

$$\mathbf{\Omega}^{-1} = \left\{\sum_{t=1}^{n}\left(\sum_{l=1}^{\kappa'} x_t^{(l)^2} + \sum_{m=1}^{\kappa} y_t^{(m)^2}\right)\right\}^{-1}\tilde{\mathbf{\Omega}} \tag{61}$$

where

$$\tilde{\mathbf{\Omega}} = \sum_{t=1}^{n}\left(\sum_{l=1}^{\kappa'} x_t^{(l)}\mathbf{X}_t^{(l)} - \sum_{m=1}^{\kappa} y_t^{(m)}\mathbf{Y}_t^{(m)}\right)$$

if

$$\begin{aligned}
\mathbf{X}_t^{(l)^2} &= \mathbf{I} && (t = 1, 2, \ldots, n;\, l = 1, 2, \ldots, \kappa'),\\
\mathbf{X}_t^{(l)}\mathbf{X}_{t'}^{(l')} + \mathbf{X}_{t'}^{(l')}\mathbf{X}_t^{(l)} &= \mathbf{0} && (t = 1, 2, \ldots, n-1;\\
& && t' = t+1, t+2, \ldots, n;\\
& && l = 1, 2, \ldots, \kappa';\, l' = 1, 2, \ldots, \kappa'),\\
\mathbf{X}_t^{(l)}\mathbf{Y}_{t'}^{(m)} &= \mathbf{Y}_{t'}^{(m)}\mathbf{X}_t^{(l)} && (t = 1, 2, \ldots, n;\, t' = 1, 2, \ldots, n; \quad (62)\\
& && l = 1, 2, \ldots, \kappa';\, m = 1, 2, \ldots, \kappa),\\
\mathbf{Y}_t^{(m)^2} &= -\mathbf{I} && (t = 1, 2, \ldots, n;\, m = 1, 2, \ldots, \kappa),\\
\mathbf{Y}_t^{(m)}\mathbf{Y}_{t'}^{(m')} + \mathbf{Y}_{t'}^{(m')}\mathbf{Y}_t^{(m)} &= 0 && (t = 1, 2, \ldots, n-1;\\
& && t' = t+1, t+2, \ldots, n;\\
& && m = 1, 2, \ldots, \kappa;\, m' = 1, 2, \ldots, \kappa).
\end{aligned}$$

Given the set $\mathbf{\Gamma}_t^{(r)}$ $(t = 1, 2, \ldots, r;\, r \geqslant n\kappa\kappa')$ of matrices which satisfy Eqs. (23), it is sufficient to take

$$\begin{aligned}
\mathbf{X}_t^{(l)} &= \mathbf{\Gamma}_{\eta_t,l}^{(r)} && (t = 1, 2, \ldots, n;\, l = 1, 2, \ldots, \kappa'),\\
\mathbf{Y}_t^{(m)} &= \mathbf{\Gamma}_1^{(r)}\Gamma_{\eta',m}^{(r)} && (t = 1, 2, \ldots, n;\, m = 1, 2, \ldots, \kappa)
\end{aligned}$$

(where the integers $\{\eta_t\}$, $\{\eta_t'\}$ are made to correspond by a suitable diagonalization procedure to the positive integers $(2, 3, \ldots)$) for Eqs. (62) to be satisfied identically. Thus the required isomorphism has been established.

In future developments of the theory, it may well be essential to distinguish between various spacelike and timelike coordinates but if, as we do in this paper, we restrict our attention to the formal requirements imposed by the vector-matrix isomorphism, it will be seen that the various components $\{x_t^{(l)}\}$ and $\{y_t^{(m)}\}$ with differing superscripts do not play essentially separate roles; these components are merely combined without distinction of superscript in the various sums occurring in formulas (60) and (61).

Thus in this paper we chose to present the theory in terms of vectors of complex numbers, rather than in terms of vectors such as (59); the use of the latter would greatly have complicated the formulation without adding substantially to the content of our results.

APPENDIX B. THE FUNCTIONAL PADÉ TABLE

In this appendix we derive a functional form of the ε algorithm and introduce the concept of a functional continued fraction.

First, we suppose that the components of the infinite-dimensional vector z are functions of a complex variable, ξ say, and furthermore that

$$\mathbf{z} = (z(0), z(h), z(2h), \ldots). \tag{63}$$

By use of the definition of the vector inverse, we then easily derive

$$(h\mathbf{z})^{-1} = \left\{\sum_{t=0}^{\infty} hz(th)\bar{z}(th)\right\}^{-1} \bar{\mathbf{z}}.$$

Assuming that the integral concerned exists, we are led immediately to the concept of a functional inverse

$$\mathbf{z}(\xi)^{-1} = \left\{\int_0^{\infty} \mathbf{z}(t)\bar{\mathbf{z}}(t)\, dt\right\}^{-1} \bar{\mathbf{z}}(\xi).$$

Second, we replace the vectors $\{\boldsymbol{\epsilon}_r^{(m)}\}$ occurring in relationships (4) by a new set $\{\boldsymbol{\epsilon}_r^{(m)*}\}$ given by

$$\boldsymbol{\epsilon}_{2r}^{(m)*} = \boldsymbol{\epsilon}_{2r}^{(m)}, \qquad \boldsymbol{\epsilon}_{2r+1}^{(m)*} = h\boldsymbol{\epsilon}_{2r+1}^{(m)} \qquad (r = 0, 1, \ldots; m = -r-1, -r, \ldots); \tag{64}$$

the vector epsilon algorithm relationships then evolve to the form

$$\boldsymbol{\epsilon}_{r+1}^{(m)*} = \boldsymbol{\epsilon}_{r-1}^{(m+1)*} + \{h(\boldsymbol{\epsilon}_r^{(m+1)*} - \boldsymbol{\epsilon}_r^{(m)*})\}^{-1}$$
$$(r = 0, 1, \ldots;\ m = -(r \div 2) - 1, -(r \div 2), \ldots).$$

Now we use the two substitutions (63) and (64) in conjunction, and let h tend to zero: every member of the ε array then becomes a function, which we denote by $\boldsymbol{\epsilon}_r^{(m)}(\xi)$ (we omit the asterisks for simplicity), and the ε algorithm relationships evolve to their functional form

$$\boldsymbol{\epsilon}_{r+1}^{(m)}(\xi) = \boldsymbol{\epsilon}_{r-1}^{(m+1)}(\xi) + \{\boldsymbol{\epsilon}_r^{(m+1)}(\xi) - \boldsymbol{\epsilon}_r^{(m)}(\xi)\}^{-1} \tag{65}$$
$$(r = 0, 1, \ldots; m = -(r \div 2) - 1, -(r \div 2), \ldots).$$

L_2 integrable functions are closed with respect to addition, subtraction, multiplication by a finite scalar variable, and what we have defined to be inversion. Thus we obtain

THEOREM 14. *We are given a set of coefficients* $\{\mathbf{u}_t(\xi)\}$, L_2 *integrable functions of the complex variable* ξ. *The functions* $\{\boldsymbol{\epsilon}_r^{(m)}(\xi)\}$ *are derived by application of relationships* (65) *to the initial values*

$$\boldsymbol{\epsilon}_{-1}^{(m)}(\xi) = 0 \quad (m = 0, 1, \ldots), \qquad \boldsymbol{\epsilon}_{2m}^{(-m-1)}(\xi) = 0 \quad (m = 0, 1, \ldots),$$
$$\boldsymbol{\epsilon}_0^{(m)}(\xi) = \sum_{t=0}^{m} u_t(\xi)\mu^t \qquad (0 < |\mu| < \infty; m = 0, 1, \ldots).$$

The only way in which the construction of the functions $\{\boldsymbol{\epsilon}_r^{(m)}(\xi)\}$ *can break down is due to the presence of one or more equal pairs of functions such as* $\boldsymbol{\epsilon}_r^{(m+1)}(\xi)$ *and* $\boldsymbol{\epsilon}_r^{(m)}(\xi)$. *Those functions* $\{\boldsymbol{\epsilon}_r^{(m)}(\xi)\}$ *of the* ε *array which can be constructed are* L_2 *functions.*

In analogy with the vector case, we may clearly regard the even order functional ε array as being a functional Padé table.

We are, however, obliged to point out that we have not derived an isomorphism between functions of a scalar variable and certain linear operators, as was done between vectors and certain matrices in the main body of this paper.

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